2.5: Matrix Factorizations

Key idea: To "factor" or "factorize" is to decompose a product into two or more constituent factors. A familiar example is the factorization of quadratic polynomials, e.g. $\underbrace{(x^2-1)}_{\text{product}} = \underbrace{(x+1)(x-1)}_{\text{factors}}$.
In this section we learn how to decompose a matrix into two factors: $\underbrace{A}_{\text{product}} = \underbrace{B \cdot C}_{\text{factors}}$.
We will only see one method in class, but there are many others, two of which you will see in the homework. The book discusses ways in which these factorizations aid in computational efficiency for applications.

In many instances, computation with (and analysis of) a matrix A can be greatly aided by decomposing A into simpler factors.

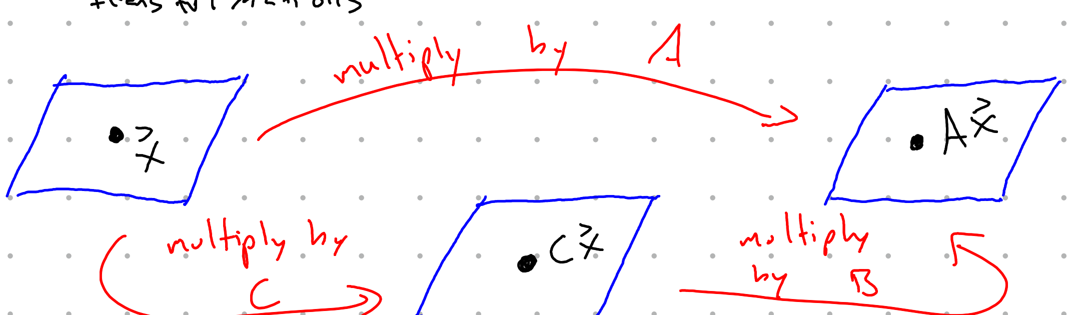
$$A = B \cdot C$$

If X and Y are simple to work with (diagonal, sparse, triangular, etc) then the question of say, finding an $\vec{x}$ s.t.

$A\vec{x} = \vec{b}$ is reduced to finding $\vec{x}$ and $\vec{y}$ s.t. $C\vec{x} = \vec{y}$, $B\vec{y} = \vec{b}$.

This is even more beneficial if we want to solve many equations of this form involving A.

Later on, we can apply this idea to the decomposition of linear transformations



The properties of B and C determine the type and utility of the factorization: here we consider only one sort of factorization in which the factors B and C are both triangular. You will consider other factorizations in the homework.

LU Factorization: $A = LU$.

Given an $m \times n$ matrix A, we wish to decompose it into an $m \times m$ lower triangular matrix L and an upper triangular $m \times n$ matrix U.

We begin with an example before proceeding to the method of determining L and U.

check this!

Ex) Notice that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ 9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ 3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

$$\underbrace{}_{A} = \underbrace{}_{L} \cdot \underbrace{}_{U}$$

Use this factorization to solve $A x = \vec{b}$ where $\vec{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

$$A\vec{x} = \vec{b} \Rightarrow LU\vec{x} = \vec{b} \Rightarrow U\vec{x} = \vec{y} \text{ and } L\vec{y} = \vec{b}$$

Find $\vec{y}$: $\begin{bmatrix} L & \vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \vec{y} = \begin{bmatrix} -9 \\ -4 \\ 5 \\ 1 \end{bmatrix}$.

$$\left(\text{So } \begin{matrix} y_1 = -9 \\ y_2 = 5 + (-9) = -4 \end{matrix} \text{ and } \begin{matrix} y_3 = 7 - 2(-9) + 5(-4) = 5 \\ y_4 = 11 + 3(-9) - 8(-4) - 3(5) = 1 \end{matrix} \right)$$

So $L\vec{y} = \vec{b}$, if $U\vec{x} = \vec{y}$ then $\vec{b} = L\vec{y} = LU\vec{x} = A\vec{x}$.

Find $\vec{x}$: $\begin{bmatrix} U & \vec{y} \end{bmatrix} = \begin{bmatrix} 3 & -7 & -2 & 2 & 9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \boxed{\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}}$.

(so $\vec{x}_4 = -1$

$x_3 = -5 + 1(-1) = -6 \ldots$)

This may seem needlessly complex, but in fact this method required 28 arithmetic operations while row reducing $[A\ \vec{b}]$ needs 62 arithmetic operations. If we were asked to solve $A\vec{x} = \vec{b}$ with say 1000 different choices of $\vec{b}$, (which is certainly common in applications), using LU would save us $1000(62-28) = 34{,}000$ arithmetic operations!!

Thus, for a matrix, determining the LU factorization is quite advantageous. Though not necessary, we will consider only matrices A which do not need row interchanges to be reduced to echelon form.

An algorithm for finding LU:

Given an $m \times n$ matrix A, find L and U by:

1) Row reduce A to an echelon form U by a sequence of row replacement operations.

2) Construct L to be $m \times m$ lower triangular with 1s along the diagonal and other entries s.t. the operations which reduce A to U, reduce L to I.

Essentially, we clear below the pivots of A to find U and keep track of our work to build L.

Ex1 Find an LU factorization of
$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}.$$

L is 4×4: U is 4×5 and an echelon form of A.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

-- fill in --

Row reduce A, the pivot columns determine L:

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$$

Notice
$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$
$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \leftarrow U$$

So we have
$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

and indeed $A = L\cdot U$. (You should check this!)